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On stochastic areas and averages of planar Brownian motion

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Abstract. Duplantier has computed the characteristic function (CF) of the stochastic area of planar Brownian motion taken with respect to its centre of gravity. In this paper it is shown that this result can be deduced from Lévy's stochastic area formula. Moreover, Duplantier's stochastic area is shown to have the same law as the stochastic area of the restriction to the time interval [0, 1] of the Brownian ring defined on the time interval $[0, \frac{4}{3}]$.

The same method also applies to the stochastic area formulae, obtained by Biane and Yor, which are relative to certain planar processes associated with the orthogonal decomposition of Brownian motion along the $L^2[0, 1]$ basis of Legendre polynomials.

1. Introduction

Denote by $\zeta \times \eta$ the quantity $\mathrm{Im}(\bar{\zeta}\eta)$, for $\zeta, \eta \in \mathbb{C}$. Let $(Z_s = X_s + \mathrm{i}Y_s, s \geqslant 0)$ be the complex-valued Brownian motion, with $Z_0 = 0$, and let $G = \int_0^1 \mathrm{d}s\, Z_s$ be its centre of gravity over the time interval [0, 1]. In the preceding article, Duplantier computed the distribution of

$$\mathscr{A}_G = \int_0^1 (Z_s - G) \times \mathrm{d}(Z_s - G)$$

after obtaining its CF (characteristic function)

$$E[\exp(\mathrm{i}\lambda \mathscr{A}_G)] = \frac{4}{\cosh \lambda + 3 \sinh \lambda / \lambda}. \tag{1.1}$$

In this paper, we show how to derive (1.1) from the celebrated Lévy (1950) formula:

$$E\left[\exp(\mathrm{i}\lambda \mathscr{A}) \,|\, Z_1 = z\right] = \left(\frac{\lambda}{\sinh \lambda}\right) \exp\left(-\frac{|z|^2}{2}(\lambda \coth^{-1} \lambda - 1)\right) \tag{1.2}$$

where $\mathscr{A} = \int_0^1 Z_s \times \mathrm{d}Z_s$. In fact, in order to understand better the roles of the constants 4 and 3 in formula (1.1) above, we shall extend (1.1) by computing the CF of

$$\mathscr{A}^{(\alpha)} \equiv \int_0^1 (Z_s - \alpha G) \times \mathrm{d}(Z_s - \alpha G)$$

for any $\alpha \in \mathbb{R}$, again with the help of (1.2). This is done in § 2.

In § 3 we remark that, when $|1 - (\alpha/2)| < 1$, the law of $\mathscr{A}^{(\alpha)}$ is that of the stochastic area of the restriction to [0, 1] of the Brownian ring (called the Brownian bridge, or Brownian lace, by probabilists) defined on the time interval $[0, a]$, for some constant $a > 1$ (in the case $\alpha = 1$, $a = \frac{4}{3}$). Section 4 is devoted to a few generalisations of the previous results, and further remarks. In particular, we discuss how the results obtained in §§ 2 and 3 are related to the orthogonal decomposition of Brownian motion along the $L^2[0,1]$ basis of Legendre polynomials, which—as shown by Biane and Yor (1986)—is intimately linked with the stochastic area of Brownian motion. Furthermore, the discussion made in § 3 for $|1-(\alpha/2)|<1$ is then extended to any $\alpha \in \mathbb{R}$.

2. Reduction to Lévy's formula (1.2)

Consider the decomposition

$$Z_s = \tilde{Z}_s + sZ_1 \qquad s \leqslant 1 \tag{2.1}$$

where $(\tilde{Z}_s = Z_s - sZ_1,\ s \leqslant 1)$ is a representation of the Brownian ring. Furthermore, since for every s we have†

$$E[\tilde{X}_s X_1] = E[\tilde{Y}_s Y_1] = 0$$

the Brownian ring $(\tilde{Z}_s,\ s \leqslant 1)$ is *independent* of the variable Z_1.

Using the decomposition (2.1), we develop $\mathscr{A} \equiv \int_0^1 Z_s \times \mathrm{d}Z_s$, and we find

$$\mathscr{A} = \tilde{\mathscr{A}} + Z_1 \times \beta \qquad \text{where } \tilde{\mathscr{A}} = \int_0^1 \tilde{Z}_s \times \mathrm{d}\tilde{Z}_s \tag{2.2}$$

$$\beta = \int_0^1 (s\,\mathrm{d}\tilde{Z}_s - \tilde{Z}_s\,\mathrm{d}s) = \int_0^1 (s\,\mathrm{d}Z_s - Z_s\,\mathrm{d}s) = Z_1 - 2\int_0^1 \mathrm{d}s\, Z_s. \tag{2.3}$$

This last expression for β is obtained from the previous one by integration by parts.

We remark that the first equality in (2.3) expresses β in terms of the process $\tilde{Z}$; therefore, since the process $(\tilde{Z}_s, s \leqslant 1)$ and the variable Z_1 are independent, we find that the three-dimensional RV $(\tilde{\mathscr{A}}, \beta)$ is independent of the variable Z_1; this fact shall play an essential role in what follows. We also note that the last equality in (2.3) may be written as

$$\beta = Z_1 - 2G. \tag{2.4}$$

We now proceed towards the computation, for any $\alpha \in \mathbb{R}$, of the CF of $\mathscr{A}^{(\alpha)}$. Before doing so, we remark, with the help of (2.4), that

$$\mathscr{A}^{(0)} = \mathscr{A} \qquad \mathscr{A}^{(1)} = \mathscr{A}_G \qquad \mathscr{A}^{(2)} = \tilde{\mathscr{A}} \tag{2.5}$$

and, in general,

$$\mathscr{A}^{(\alpha)} = \tilde{\mathscr{A}} + (1-\alpha/2) Z_1 \times \beta = \mathscr{A} - (\alpha/2) Z_1 \times \beta. \tag{2.6}$$

We now prove that, for any $\lambda \in \mathbb{R}$, and $z \in \mathbb{C}$,

$$E[\exp(\mathrm{i}\lambda \mathscr{A}^{(\alpha)}) \mid Z_1 = z] = E[\exp(\mathrm{i}\lambda \mathscr{A}) \mid Z_1 = (1-\alpha/2)z]. \tag{2.7}$$

Indeed, we have, on one hand, from (2.6),

$$E[\exp(\mathrm{i}\lambda \mathscr{A}^{(\alpha)}) \mid Z_1 = z] = E[\exp \mathrm{i}\lambda(\tilde{\mathscr{A}} + (1-\alpha/2) z \times \beta)]$$

using the fact, already pointed out before (2.4), that the pair $(\tilde{\mathscr{A}}, \beta)$ is independent of the variable Z_1.

† This is due to the linearity of the expectation, and the fact that $E[X_s X_t] = E[Y_s Y_t] = \min(s, t)$.

On the other hand, we deduce from (2.2) that

$$E[\exp(\mathrm{i}\lambda\mathscr{A})\,|\,Z_1=(1-\alpha/2)z]=E[\exp \mathrm{i}\lambda(\tilde{\mathscr{A}}+(1-\alpha/2)z\times\beta)]$$

and we have proved (2.7).

The computation of the CF of $\mathscr{A}^{(\alpha)}$ now follows immediately from Lévy's formula (1.2) and formula (2.7) by integrating in z with respect to the law of Z_1. We obtain

$$E[\exp(\mathrm{i}\lambda\mathscr{A}^{(\alpha)})]=[(1-\alpha/2)^2\cosh\lambda+(\alpha/2)(2-\alpha/2)\sinh\lambda/\lambda]^{-1}. \tag{2.8}$$

In particular, taking respectively $\alpha=0,\ 2,\ 1$, we obtain, from (2.5),

$$E[\exp(\mathrm{i}\lambda\mathscr{A})]=1/\cosh\lambda \qquad E[\exp(\mathrm{i}\lambda\tilde{\mathscr{A}})]=\lambda/\sinh\lambda$$
$$E[\exp(\mathrm{i}\lambda\mathscr{A}_G)]=\frac{4}{\cosh\lambda+3\sinh\lambda/\lambda}. \tag{2.9}$$

In the preceding article, Duplantier shows how to obtain these formulae from functional integrals.

3. Relationship with restrictions of Brownian rings

The following formula:

$$\mathscr{A}^{(\alpha)}=\int_0^1\left(Z_u-\frac{\alpha}{2}uZ_1\right)\times\mathrm{d}\left(Z_u-\frac{\alpha}{2}uZ_1\right) \tag{3.1}$$

is easily deduced from (2.6), since the right-hand side of (3.1) is equal to

$$\mathscr{A}-(\alpha/2)Z_1\times\beta.$$

We shall now prove that, in the case $|1-\alpha/2|<1$, the process $(Z_u-(\alpha/2)uZ_1;\ u\leqslant 1)$ is distributed as the restriction to $[0,1]$ of the Brownian ring on $[0,a]$, with a defined by

$$(1-\alpha/2)^2=1-1/a \tag{3.2}$$

(in the case $\alpha=1$, we find $a=\frac{4}{3}$).

Indeed, a representation of the Brownian ring on $[0,a]$ is

$$(Z_u-(u/a)Z_a,\ u\leqslant a)$$

and we have, on one hand, with the notation $\tilde{Z}_u=Z_u-uZ_1$,

$$Z_u-(\alpha/2)uZ_1=\tilde{Z}_u+u(1-\alpha/2)Z_1 \qquad u\leqslant 1$$

and, on the other hand,

$$Z_u-(u/a)Z_a=\tilde{Z}_u+u(Z_1-(1/a)Z_a) \qquad u\leqslant 1.$$

Since the process $(\tilde{Z}_u,\ u\leqslant 1)$ is independent from $(Z_t,\ t\geqslant 1)$, the two processes $(Z_u-(\alpha/2)uZ_1,\ u\leqslant 1)$ and $(Z_u-(u/a)Z_a,\ u\leqslant 1)$ will have the same law once the Gaussian variables $(1-\alpha/2)Z_1$ and $(Z_1-(1/a)Z_a)$ have the same distribution; this is so if and only if (3.2) is satisfied. Hence we have shown, under the condition $|1-\alpha/2|<1$, that the distribution of $\mathscr{A}^{(\alpha)}$ is that of the stochastic area of the restriction to $[0,1]$ of the Brownian ring on $[0,a]$, with a defined by (3.2).

4. Generalisations and concluding remarks

There are a number of points we would like to make.

(i) The key formula (2.7) which allows one to reduce the computation of the CF of $\mathscr{A}^{(\alpha)}$ to that of the conditional CF of $\mathscr{A}$ given Z_1 is valid, *mutatis mutandis*, for a large class of complex-valued processes $(Z_s = X_s - \mathrm{i}\, Y_s,\ s \geqslant 0)$. Indeed, it suffices that X and Y are continuous Gaussian processes, centred (i.e. they have mean 0, for every s), independent, have the same law and, moreover, are semi-martingales†.

Then, the decomposition (2.1) is altered into

$$Z_s = \tilde{Z}_s + \varphi(s) Z_1 \tag{4.1}$$

where $\varphi(s)$ is defined by

$$E[X_1^2]\varphi(s) = E[X_s X_1] \tag{4.2}$$

and G is defined by

$$G = \int_0^1 \mathrm{d}\varphi(s) Z_s \tag{4.3}$$

(the semi-martingale hypothesis made on Z automatically implies that φ has bounded variation; see Emery (1982)).

After these changes have been made, the formulae (2.4), (2.6) and (2.7) are still valid.

Particularly interesting cases are obtained with the Gaussian processes $Z = V_p$, where, for $p > -\frac{1}{2}$, one defines

$$V_p(t) = \frac{1}{t^p} \int_0^t \mathrm{d}B_s\, s^p \qquad t > 0$$

in terms of the complex Brownian motion $(B_s, s \geqslant 0)$. The decomposition (4.1) now holds with $\varphi_p(s) = s^{p+1}$, and (4.3) becomes

$$G_p = (p+1) \int_0^1 \mathrm{d}s\, s^p V_p(s) = (p+1) \int_0^1 \mathrm{d}B_u\, u^p (1-u).$$

Using now the notation $\mathscr{A}_p$ and $\mathscr{A}_p^{(\alpha)}$ instead of $\mathscr{A}$ and $\mathscr{A}^{(\alpha)}$, we recall that Biane and Yor (1986) have obtained the following extension of Lévy's formula (1.2):

$$E[\exp(\mathrm{i}\lambda \mathscr{A}_p) \mid V_p(1) = z] = a_p(|\lambda|) \exp(-(|z|^2/2) b_p(|\lambda|)) \tag{4.4}$$

where

$$a_p(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1) I_\nu(x)} \qquad b_p(x) = x \frac{I_{\nu+1}(x)}{I_\nu(x)} \qquad \nu = p + \tfrac{1}{2} \qquad x > 0 \tag{4.5}$$

and I_ν is the modified Bessel function of index ν.

We now obtain the following extension (4.6) of formula (2.8), which is deduced from (2.7) and (4.4):

$$E[\exp(\mathrm{i}\lambda \mathscr{A}_p^{(\alpha)})] = \frac{1}{2^\nu \Gamma(\nu+1)} \frac{|\lambda|^\nu}{I_\nu(|\lambda|) + (|\lambda|/2\nu)(1-\alpha/2)^2 I_{\nu+1}(|\lambda|)}. \tag{4.6}$$

† This technical constraint—which we will not discuss here—has been at the heart of the developments and understanding of stochastic integration for the past thirty years; see, for example, Rogers and Williams (1987).

(ii) The interest of the author in the computation (1.1) of the CF of $\mathscr{A}_G$, made by Duplantier in the preceding article, was originally aroused by the following connection with the work of Biane and Yor (1986): these authors developed the Brownian path $(Z_t, t \leq 1)$ along the $L^2[0, 1]$ basis of Legendre polynomials. Precisely:

$$Z_t = \sum_{p=0}^{\infty} u_{p+1}(t)\beta_p \tag{4.7}$$

where

$$u_{p+1}(t) = (2p+1)\int_0^t \mathrm{d}s\, P_p(2s-1) \qquad \beta_p = \int_0^1 \mathrm{d}Z_s\, P_p(2s-1)$$

and $(P_p\,; p = 0, 1, 2, \ldots)$ is the sequence of Legendre polynomials, which are classically defined by

$$P_p(t) = \frac{1}{2^p p!}\frac{\mathrm{d}^p}{\mathrm{d}t^p}((t^2-1)^p).$$

In particular, we find

$$\beta_0 = Z_1 \qquad \text{and} \qquad \beta_1 = \int_0^1 \mathrm{d}Z_s(2s-1) = Z_1 - 2G \tag{4.8}$$

so that β_1 is the variable β featured in (2.4) above.

Moreover, the stochastic area $\mathscr{A}$ has a simple representation in terms of the variables β_p, namely

$$\mathscr{A} = \sum_{p=0}^{\infty} \beta_p \times \beta_{p+1} \tag{4.9}$$

where the convergence holds both in L^2 and almost surely.

Furthermore, for any $p \in \mathbb{N}$, Biane and Yor (1986) showed

$$E[\exp(\mathrm{i}\lambda\mathscr{A})|\beta_k = m_k\,; 0 \leq k \leq p] = \exp\left(\mathrm{i}\lambda \sum_{k=0}^{p-1} m_k \times m_{k+1}\right) a_p(|\lambda|)\exp\left(-\frac{|m_p|^2}{2}b_p(|\lambda|)\right) \tag{4.10}$$

where a_p and b_p are defined by (4.5).

Consequently, making use of the identification (4.8), and of (2.6) which now becomes

$$\mathscr{A}^{(\alpha)} = \mathscr{A} - (\alpha/2)\beta_0 \times \beta_1 \tag{4.11}$$

our formula (2.8), which generalises Duplantier's formula (1.1), can be deduced from the result (4.10) above taken for $p = 1$; in this case the functions a_1 and b_1 are

$$a_1(x) = \frac{x^3}{3(x\cosh x - \sinh x)} \qquad b_1(x) = x^2\left(\frac{\sinh x}{x\cosh x - \sinh x}\right) - 3.$$

Now, in order to obtain (2.8) it remains, using (4.10) and (4.11), to integrate

$$\exp\{\mathrm{i}\lambda(1-\alpha/2)m_0 \times m_1\}a_1(\lambda)\exp\{-(|m_1|^2/2)b_1(\lambda)\}$$

with respect to the law $P(\beta_0 \in \mathrm{d}m_0, \beta_1 \in \mathrm{d}m_1)$ of the two independent, centred, Gaussian variables β_0 and β_1 with variances:

$$\tfrac{1}{2}E(|\beta_0|^2) = 1 \qquad \tfrac{1}{2}E(|\beta_1|^2) = \tfrac{1}{3}.$$

Thus, we obtain

$$E[\exp(\mathrm{i}\lambda\mathscr{A}^{(\alpha)})] = a_1(\lambda)E[\exp\{-(\lambda^2/2)(1-\alpha/2)^2|\beta_1|^2 - (b_1(\lambda)/2)|\beta_1|^2\}]$$

and, since $\frac{1}{2}|\beta_1|^2 = \frac{1}{3}H$, where H is an exponential variable (that is: $P(H \in \mathrm{d}h) = e^{-h}\,\mathrm{d}h$), the above formula yields

$$E[\exp(\mathrm{i}\lambda\mathscr{A}^{(\alpha)})] = \frac{a_1(\lambda)}{1+\frac{1}{3}[\lambda^2(1-\alpha/2)^2 + b_1(\lambda)]}$$

which gives (2.8).

However, the purpose of § 2 above has been to simplify this approach by going back to Lévy's original formula (1.2) rather than relying upon its extension (4.10).

Nonetheless, it might be of some interest to extend the definition of $\mathscr{A}_G$ and $\mathscr{A}^{(\alpha)}$ further in a way which would relate with formula (4.10) for a general integer p.

Formula (3.1) suggests such an extension. Indeed, starting from the development (4.7) of $(Z_t, t \leqslant 1)$, we introduce

$$Z_t^c = \sum_{p=0}^{\infty} c_p u_{p+1}(t)\beta_p$$

where the sequence $\boldsymbol{c} = (c_p,\ p \in \mathbb{N})$ consists of real numbers which are all, except for some finite set, equal to 1. We then define

$$\mathscr{A}^c \equiv \int_0^1 Z_s^c \times \mathrm{d}Z_s^c$$

(from (3.1), $\mathscr{A}^{(\alpha)}$ corresponds to $c_0 = 1-\alpha/2$, and $c_p \equiv 1$, for $p \geqslant 1$) and, using the development of Z_t^c in terms of the sequence (β_p), we obtain the following modification of (4.9):

$$\mathscr{A}^c = \sum_{k=0}^{\infty} c_k c_{k+1}\beta_k \times \beta_{k+1}.$$

If p is an integer such that for any $k \geqslant p$, $c_k = 1$, we deduce from (4.10) that

$$E[\exp(\mathrm{i}\lambda\mathscr{A}^c)] = E\left[\exp\left(\mathrm{i}\lambda\sum_{k=0}^{p-1} c_k c_{k+1}\beta_k \times \beta_{k+1}\right)\exp\left(-\frac{|\beta_p|^2}{2}b_p(|\lambda|)\right)\right].$$

Making use of the variance formulae (see Biane and Yor (1986))

$$\tfrac{1}{2}E(|\beta_k|^2) = \frac{1}{2k+1}$$

and of the independence of the β_k, we can write:

$$E[\exp(\mathrm{i}\lambda\mathscr{A}^c)] = a_p(|\lambda|)E\left[\exp\left(\mathrm{i}\sum_{k=0}^{p} \mathrm{d}_k\,\gamma_k \times \gamma_{k+1}\right)\right]$$

where the γ_k are independent, reduced, complex-valued Gaussian variables, and the vector $\boldsymbol{d} = (d_k\,;\, 0 \leqslant k \leqslant p)$ is defined by

$$d_k = \lambda c_k c_{k+1}((2k+1)(2k+3))^{-1/2} \qquad 0 \leqslant k \leqslant p-1$$

$$d_p = (2p+1)^{-1}b_p(|\lambda|).$$

It now follows from theorem 2 of Biane and Yor (1987) that

$$E[\exp(\mathrm{i}\lambda \mathscr{A}^c)] = a_p(|\lambda|) \prod_{k=0}^{p} C_k(\boldsymbol{d})$$

where

$$C_k(\boldsymbol{d}) = \cfrac{1}{1+\cfrac{d_{k+1}^2}{1+\cfrac{d_{k+2}^2}{1+\cfrac{\ddots}{1+d_p^2}}}}.$$

(iii) We now make some comments on § 3.

We first explain, with the help of (3.1), why, for any $\alpha \in \mathbb{R}$, $\mathscr{A}^{(\alpha)}$ and $\mathscr{A}^{(4-\alpha)}$ have the same law, as shown by (2.8). This follows from the identity in law of the two processes $(Z_u - (\alpha/2)uZ_1,\ 0 \leqslant u \leqslant 1)$ and $(Z_u - [(4-\alpha)/2]uZ_1,\ 0 \leqslant u \leqslant 1)$, which follows from the decomposition

$$Z_u - (\alpha/2)uZ_1 = \tilde{Z}_u + (1-\alpha/2)uZ_1 \qquad u \leqslant 1$$

the independence of $(\tilde{Z}_u,\ u \leqslant 1)$ and Z_1, and the fact that Z_1 and $-Z_1$ have the same (Gaussian) law.

In the particular case $\alpha = 0$, this shows that $(Z_u - 2uZ_1,\ u \leqslant 1)$ has the same law as $(Z_u,\ u \leqslant 1)$, and hence is a Brownian motion.

Next we remark that, although formula (3.1) is valid for any $\alpha \in \mathbb{R}$, we presented an interpretation of this formula, hence of (2.8) for the CF of $\mathscr{A}^{(\alpha)}$, in terms of Brownian rings, only in the case $|1-\alpha/2| < 1$. In fact, a very different interpretation is needed when $|1-\alpha/2| \geqslant 1$.

We have just explained why in the case $|1-\alpha/2| = 1$, which implies $\alpha = 0$ or $\alpha = 4$, $\mathscr{A}^{(4)}$ and $\mathscr{A}$ have the same law. To complete the discussion, we now make some remarks about the process $(Z_u - (\alpha/2)uZ_1,\ u \leqslant 1)$, when $|1-\alpha/2| > 1$. We first recall several representations of the standard Brownian ring. If $(\zeta(t),\ t \leqslant 1)$ is a one-dimensional Brownian motion, starting from 0, then the following processes, indexed by $t \in \,]0, 1[$,

$$(1-t)\zeta\left(\frac{t}{1-t}\right) \qquad t\zeta\left(\frac{1}{t}-1\right) \qquad \zeta(t) - t\zeta(1) \tag{4.12}$$

are three (different) representations of the Brownian ring, the last of which was constantly used above. Changing − into + in the first expression in (4.12), we define the three following Gaussian processes indexed by $t > 0$:

$$(1+t)\zeta\left(\frac{t}{1+t}\right) \qquad t\zeta\left(\frac{1}{t}+1\right) \qquad \eta(t) + t\zeta(1) \tag{4.13}$$

where η is a second Brownian motion starting from 0, and independent of ζ; all have the same law.

The assertion concerning (4.12) is well known, but we remark that, using the notation ζ, $\hat{\zeta}$ and $\tilde{\zeta}$ for Brownian motion, we pass from the first expression in (4.12) to the second by representing† $\zeta(u)$ as $u\hat{\zeta}(\frac{1}{u})$; then

$$(1-t)\zeta\left(\frac{t}{1-t}\right)=t\hat{\zeta}\left(\frac{1}{t}-1\right)$$

and we pass from the second expression in (4.12) to the third by representing† $\zeta(u)$ as $\tilde{\zeta}(1+u)-\tilde{\zeta}(1)$, and then $\tilde{\zeta}(u)$ as $u\hat{\zeta}(1/u)$. The same transformations give the identity in law between the three processes featured in (4.13).

We shall use the notation $(\theta^-(t),\ 0\leqslant t\leqslant 1)$, respectively $(\theta^+(t),\ t>0)$, for any process whose law is that of the Brownian ring, respectively for any process whose law is that of any of the three processes featured in (4.13). The distributions of these two centred Gaussian processes may be characterised by their covariances:

$$E[\theta^-(s)\theta^-(t)]=s(1-t)\qquad E[\theta^+(s)\theta^+(t)]=s(1+t)\qquad s<t. \tag{4.14}$$

These formulae are deduced from any of the representations in (4.12) and (4.13), and the well known covariance, $\min(s,t)$, of Brownian motion.

Furthermore, we define, for $0<c<1$, respectively for $c>0$:

$$\theta_c^-(t)=\frac{1}{\sqrt{c}}\,\theta^-(ct)\qquad t\leqslant 1$$

respectively

$$\theta_c^+(t)=\frac{1}{\sqrt{c}}\,\theta^+(ct)\qquad t\geqslant 0.$$

We can now end the discussion begun in § 3 by making the following statements.

If $|1-\alpha/2|<1$, and c is defined by $(1-\alpha/2)^2=1-c$, then the law of $(Z_u-(\alpha/2)uZ_1;\ 0\leqslant u\leqslant 1)$ is that of $(\theta_c^-(u);\ 0\leqslant u\leqslant 1)$ (which is also the restriction to [0, 1] of the Brownian ring defined on $[0, 1/c]$).

If $|1-\alpha/2|>1$, and c is defined by $(1-\alpha/2)^2=1+c$, then the law of $(Z_u-(\alpha/2)uZ_1;\ 0\leqslant u\leqslant 1)$ is that of $(\theta_c^+(u);\ 0\leqslant u\leqslant 1)$.

(iv) In conclusion, we have shown in this paper how Duplantier's computation is related to Lévy's stochastic area formula (1.2) and to the more recent works of Biane and Yor. Showing these connections enabled us to obtain a large variety of extensions of formula (1.1).

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† We use two well known invariance properties of the Brownian law: if $(\zeta(u),\ u\geqslant 0)$ is a Brownian motion, so are $(u\zeta(1/u),\ u>0)$ and $(\zeta(u+1)-\zeta(1),\ u\geqslant 0)$.

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